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Q.1. State and prove Higher Logarithmic Test.

Soln. Statement: → The series $\sum u_n$ of positive terms is convergent or divergent according as

$$\lim_{n\to\infty}\left[\left(n\log\frac{u_n}{u_{n+1}}-1\right)\log n\right] > \text{ or } < 1$$

Proof: Let us compare the given series $\sum u_n$ with the auxiliary series $\sum v_n$, where $v_n = \frac{1}{n(\log n)^p}$

We know that $\sum v_n$ is convergent or divergent according as $p >$ or < 1.

Hence, by comparison test, the given series $\sum u_n$ will be convergent or divergent according as

$$\frac{u_n}{u_{n+1}} > \text{ or } < \frac{v_n}{v_{n+1}}$$

or, $$\frac{u_n}{u_{n+1}} > \text{ or } < \frac{\frac{1}{n(\log n)^p}}{\frac{1}{(n+1)(\log(n+1))^p}}$$

or, $$\frac{u_n}{u_{n+1}} > \text{ or } < \frac{n+1}{n}\left\{\frac{\log(n+1)}{\log n}\right\}^p$$

or, $$\frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)\left[\frac{\log\left\{n\left(1+\frac{1}{n}\right)\right\}}{\log n}\right]^p$$

or, $$\frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)\left\{\frac{\log n+\log\left(1+\frac{1}{n}\right)}{\log n}\right\}^p$$

or, $$\frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)\left(\frac{\log n+\frac{1}{n}-\frac{1}{2n^2}+\cdots\text{ to }\infty}{\log n}\right)^p$$

or, $$\frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n\log n}-\frac{1}{2n^2\log n}+\cdots\text{ to }\infty\right)^p$$

or, $$\frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)\left(1+\frac{p}{n\log n}-\cdots\text{ to }\infty\right)$$

or, $$\frac{u_n}{u_{n+1}} > \text{ or } < 1+\frac{1}{n}+\frac{p}{n\log n}+\frac{p}{n^2\log n}+\cdots\text{ to }\infty.$$

or, $\log \frac{u_n}{u_{n+1}} >$ or $< \log \left[1 + \left(\frac{1}{n} + \frac{p}{n\log n} + \frac{p}{n^2 \log n} + \cdots \text{ to } \infty\right)\right]$

or, $\log \frac{u_n}{u_{n+1}} >$ or $< \frac{1}{n} + \frac{p}{n\log n} + \frac{p}{n^2\log n} + \cdots \text{ to } \infty$

or, $n\log \frac{u_n}{u_{n+1}} >$ or $< 1 + \frac{p}{\log n} + \frac{p}{n\log n} + \cdots \text{ to } \infty$

or, $n\log \frac{u_n}{u_{n+1}} - 1 >$ or $< \frac{p}{\log n} + \frac{p}{n\log n} + \cdots \text{ to } \infty$

or, $\left(n\log \frac{u_n}{u_{n+1}} - 1\right)\log n >$ or $< p + \frac{p}{n} + \cdots \text{ to } \infty$

or, $\lim_{n\to\infty} \left[\left(n\log \frac{u_n}{u_{n+1}} - 1\right)\log n\right] >$ or $< p$

or $\lim_{n\to\infty} \left[\left(n\log \frac{u_n}{u_{n+1}} - 1\right)\log n\right] >$ or < 1.

proved.

Q.2. Determine the convergency or divergency of the following series.

$$1^p + \left(\frac{1}{2}\right)^p + \left(\frac{1\cdot 3}{2\cdot 4}\right)^p + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^p + \cdots \text{ to } \infty.$$

Proof: Let $u_n = \left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\right)^p$ [omitting the first term]

and $u_{n+1} = \left(\frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{2\cdot 4\cdot 6\cdots 2n(2n+2)}\right)^p$

$\therefore \frac{u_n}{u_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^p = \left(\frac{2+\frac{2}{n}}{2+\frac{1}{n}}\right)^p$ or, $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$

Hence, D'Alembert's ratio test is helpless to suggest any result. Now, logarithmic ratio test will be helpful.

We have, $\log \frac{u_n}{u_{n+1}} = \log \frac{(1+\frac{1}{n})^p}{(1+\frac{1}{2n})^p} = p\log\left(1+\frac{1}{n}\right) - p\log\left(1+\frac{1}{2n}\right)$

$= p\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots \text{ to } \infty - \frac{1}{2n} + \frac{1}{8n^2} - \frac{1}{24n^3} + \cdots \text{ to } \infty\right]$

or, $n\log \frac{u_n}{u_{n+1}} = p\left[\frac{1}{2} - \frac{3}{8n} + \frac{7}{24n^2} - \cdots \text{ to } \infty\right]$

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Rest part of Q.2.

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or, $\lim_{n\to\infty}\left(n\log\frac{4n}{4n+1}\right)=\frac{p}{2}$.

Hence, logarithmic ratio test forces us to accept that the given series is convergent or divergent according as $\frac{p}{2}>$ or <1, that is, $p>$ or <2.

When $p=2$, then logarithmic ratio test is helpless to draw any conclussion.

Now, higher logarithmic ratio test will be helpful.

We have, $n\log\frac{4n}{4n+1}=1-\frac{3}{4n}+\frac{7}{12n^2}\cdots$ to ∞.

or, $n\log\frac{4n}{4n+1}-1=-\frac{3}{4n}+\frac{7}{12n^2}-\cdots$ to ∞.

or, $\left(n\log\frac{4n}{4n+1}-1\right)\log n=\left(-\frac{3}{4}+\frac{7}{12n}-\cdots\text{ to }\infty\right)\frac{\log n}{n}$.

or, $\lim_{n\to\infty}\left[\left(n\log\frac{4n}{4n+1}-1\right)\log n\right]=0<1$, as $\frac{\log n}{n}\to 0$ as $n\to\infty$.

Hence, higher logarithmic ratio test forces to accept that the given series is divergent when $p=2$.

Solved